

Cosmic infinity: A dynamical system approach

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Abstract

Dynamical system techniques are extremely useful to study cosmology. It turns out that in most of the cases, we deal with finite isolated fixed points corresponding to a given cosmological epoch. However, it is equally important to analyse the asymptotic behaviour of the universe. On this paper, we show how this can be carried out for 3-forms model. In fact, we show that there are fixed points at infinity mainly by introducing appropriate compactifications and defining a new time variable that washes away any potential divergence of the system. The richness of 3-form models allows us as well to identify normally hyperbolic non-isolated fixed points.

Keywords: 3-form cosmology, dynamical system, fixed points at infinity, compactification

1. Introduction

In recent years, p-form cosmology [1–4] has been the focus of a renewed interest as possible alternatives to scalar fields in explaining the nature of early inflation and the current dark energy (DE) fuelled acceleration. In particular, 3-forms, which in 4-dimensional space-times have one degree of freedom, have attracted a lot of attention since they can achieve a de Sitter phase without the need of slow-roll conditions [5].

Dynamical systems have long since been a very useful tool in cosmology [6, 7], in particular, in studying the evolution of cosmological models with scalar fields where explicit solutions of the evolution equations cannot usually be obtained. In the case of 3-forms, since the early work by Koivisto and Nunes [3], a dynamical system approach has been employed to study cosmological models with 3-forms. This includes works on 3-form inflation [8, 9] and re-heating [10], models with interaction between cold dark matter (CDM) and a 3-form field playing the role of DE [11–14], or 3-form fields in a Randall-Sundrum II braneworld scenario [15].

In systems where the dynamical variables are unbounded, a compactification scheme [16–20], which

maps the system to a compact space, is required in order to understand the flow of the system in the vicinity of infinity. In Refs. [12, 14] a compact dynamical system description for models with CDM and 3-form DE was employed. This method allowed an intuitive identification of the fixed points of the system at infinite values of the 3-form field [14]. Among these, two repulsive solutions were found, which correspond to matter dominated epochs in the asymptotic past of the system. On the present paper, we propose an adequate mathematical compactification that indeed proves that our intuitive results for those two points are correct [14]. This new compactification not only makes the 3-form compactified variable finite, as was already the case in [14], but also allows us to obtain proper finite eigenvalues for the fixed points at infinite values of the 3-form field. The method used, developed in [19], consists in parametrising the independent variable in a certain direction preserving way and then considering a certain compactification chosen from a set of admissible compactifications. This procedure maps finite critical points to finite critical points and allows us to define critical point at infinity in a way that is independent of the parametrisation. This method can be extremely important when willing to characterise the asymptotic behaviour of the Universe and one of the intrinsic degrees of freedom reaches extremely large values, making a dynamical system analysis subtle unless proper care is

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taken on the process.

This paper is structured as follows: in Sec. 2 we review the cosmological model of a homogeneous and isotropic universe filled with CDM and a 3-form field playing the role of DE and its usual dynamical system description. In Sec. 3 we review how to apply a compactification scheme to properly identify the fixed points at infinity and introduce a new dynamical system description of the model with different compact variables. In Sec. 4 we apply this new description to the particular case of a 3-form with a Gaussian potential and identify and characterise the fixed points at infinite values of the 3-form field. In the concluding Sec. 5, we discuss the results obtained and draw conclusions on the strengths and limitations of the method developed, as well as the possibility to extend it to a more general class of models.

2. 3-Form cosmology

2.1. Cosmological model

Let us consider the Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \quad (1)$$

which describes a spatially flat, homogeneous and isotropic Universe. In Eq. (1), t is the cosmic time, $a(t)$ is the scale factor and x^i , $i = 1, 2, 3$, are the comoving spatial coordinates. In such geometry the only dynamical component of a massive 3-form $A_{\mu\nu\rho}$ [2] is the spatial-like component A_{ijk} , which can be parametrised in terms of a scalar quantity $\chi(t)$ as [2–4]

$$A_{ijk} = a^3(t)\chi(t)\epsilon_{ijk}. \quad (2)$$

Here, the symbol ϵ_{ijk} is +1 (−1) if $\{ijk\}$ is an even (odd) permutation of $\{123\}$ and 0 if otherwise. In the action introduced in [2], the potential of the 3-form is an arbitrary function $V(A_{\mu\nu\rho}A^{\mu\nu\rho})$. With the ansatz (2) we can write $A_{\mu\nu\rho}A^{\mu\nu\rho} = 6\chi^2$, which means that the potential can be written as $V(\chi^2)$. The evolution equation of $\chi(t)$ reads [2–4]

$$\ddot{\chi} + 3H\dot{\chi} + 2\chi\frac{\partial V}{\partial\chi^2} = 0, \quad (3)$$

where $H := \dot{a}/a$ is the Hubble parameter.

In this work, we consider a cosmological model for the late-time Universe with a 3-form field, playing the

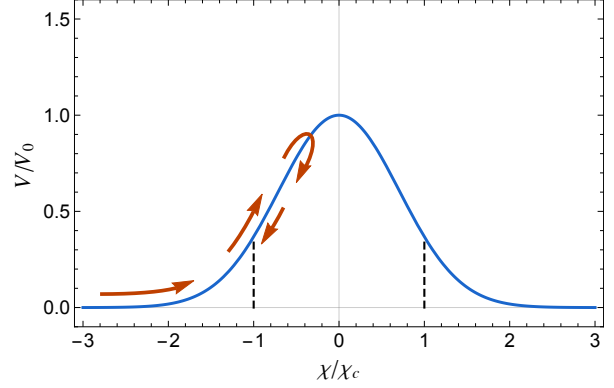


Figure 1: Representation of the evolution of the scalar quantity χ . Independently of the shape of the potential, the field χ decays until it enters the interval $[-\chi_c, \chi_c]$. Once inside, the field χ evolves towards a minimum of the potential. In the case of the Gaussian potential $V = V_0 \exp[-\xi(\chi/\chi_c)^2]$ represented here for $\xi = 1$, with V_0 a positive constant, the field decays to the value $\chi = \chi_c$ which acts as a local minimum as the field cannot escape the interval $[-\chi_c, \chi_c]$.

role of DE, and CDM. The Friedmann and Raychaudhuri equations read

$$H^2 = \frac{\kappa^2}{3} \left[\rho_m + \frac{1}{2} (\dot{\chi} + 3H\chi)^2 + V \right], \quad (4)$$

$$\dot{H} = -\frac{\kappa^2}{2} \left(\rho_m + 2\chi^2 \frac{\partial V}{\partial\chi^2} \right), \quad (5)$$

where $\kappa^2 = 8\pi G$, G is the gravitational constant, and ρ_m is the energy density of CDM, which satisfies the usual conservation equation $\dot{\rho}_m + 3H\rho_m = 0$.

As discussed in Ref. [14] a particularity of cosmological models with 3-forms is that, independently of the shape of the potential, the field χ decays monotonically until it enters the interval $[-\chi_c, \chi_c]$, with $\chi_c := \sqrt{2/(3\kappa^2)}$. Once inside, it evolves towards a minimum of the potential; since the field cannot escape this interval (cf. Figure 1), the limiting points $\chi = \pm\chi_c$ can act as local extrema. In particular, if $\partial V/\partial\chi^2(\pm\chi_c) < 0$, the limiting points can act as local attractors that lead the system towards a Little Sibling of the Big Rip (LSBR) event [21] in the asymptotic future [14], i.e., an event at infinite cosmic time where a and H blow up while \dot{H} remains finite.

2.2. Dynamical system description

A dynamical systems approach was first employed in the context of 3-form cosmology in Ref. [3]. There, three compact variables related to the fractional energy density of CDM and to the fractional kinetic and poten-

tial energy densities of the 3-form were defined ¹

$$s := \sqrt{\frac{\kappa^2 \rho_m}{3H^2}}, \quad y := \frac{\dot{\chi} + 3H\chi}{3H\chi_c}, \quad z := \sqrt{\frac{\kappa^2 V}{3H^2}}, \quad (6)$$

with $0 \leq s \leq 1$, $-1 \leq y \leq 1$, and $0 \leq z \leq 1$. With these definitions, the Friedmann equation (4) can be rewritten as

$$1 = s^2 + y^2 + z^2, \quad (7)$$

which allows the elimination of one dynamical variable from the system. An autonomous dynamical system can be obtained by defining a fourth variable χ

$$x := \frac{\chi}{\chi_c}, \quad (8)$$

where we employ a slightly different rescaling from the one used in Ref. [3] so that the critical values $\chi = \pm\chi_c$ correspond to $x = \pm 1$. Using the variables (6) and (8) we can write the evolution equations for a cosmological model with CDM and a 3-form field as

$$x' = 3(y - x), \quad (9)$$

$$y' = \frac{1}{2} [3y(1 - y^2 - z^2) + (1 - xy)\lambda(x)z^2], \quad (10)$$

$$z' = \frac{1}{2} z [3(1 - y^2 - z^2) - (y - x + xz^2)\lambda(x)], \quad (11)$$

$$s = \sqrt{1 - y^2 - z^2}. \quad (12)$$

Here, a prime indicates a derivative with respect to $N := \log(a/a_0)$ and the factor $\lambda(x)$ is defined as

$$\lambda(x) := -\sqrt{\frac{6}{\kappa^2}} \frac{1}{V} \frac{\partial V}{\partial \chi} = -3 \frac{1}{V} \frac{\partial V}{\partial x}. \quad (13)$$

The set of Eqs. (9)-(13) defines the evolution of the system in a subspace \mathcal{M} of the three-dimensional space (x, y, z) that corresponds to a half-cylinder of radius unity and infinite height: $-1 \leq y \leq 1$, $0 \leq z \leq \sqrt{1 - y^2}$, and $-\infty < x < +\infty$. The borders of the half-cylinder

$$\begin{aligned} \mathcal{M}_0 &= \{(x, y, z) \in \mathbb{R}^3 : z = 0 \wedge -1 \leq y \leq 1\}, \\ \mathcal{M}_1 &= \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 = 1 \wedge z \geq 0\}, \end{aligned} \quad (14)$$

represent two invariant subsets of the system: the plane \mathcal{M}_0 corresponds to a model with CDM and a massless 3-form, which has been found to be equivalent to

Λ CDM [3]; the surface \mathcal{M}_1 represents a universe filled solely by a 3-form field [8].

The fixed points of the system (9)-(13) were classified in Ref. [3] into three different categories: a saddle point A with $(x, y, z) = (0, 0, 0)$ that corresponds to an unstable matter era; two points B with $(x, y, z) = (\pm 1, \pm 1, 0)$ that for some choices of the potential represent late-time attractors that lead the Universe to a LSBR event [14]; a set of points C corresponding to local extrema of the potential within the interval $[-\chi_c, \chi_c]$ and which can be either attractors or saddle points. In Refs. [12, 14] the variable x was replaced by the compact variable ²

$$u := \frac{2}{\pi} \arctan(x), \quad (15)$$

with $-1 \leq u \leq 1$. This substitution allowed for the identification of the fixed points at infinite values of the field χ [14], i.e. for $u = \pm 1$. Some of the new fixed points at infinite χ were found to correspond to the asymptotic past of the system. These points are characterised by their extremely repulsive nature [14].

3. New compact description

3.1. Compactification

When a compactification scheme is employed, like the one in (15), some terms may appear in the new evolution equations that diverge as the old variables approach infinity. When this happens, the dynamical system obtained after the compactification can be written as [16–20]

$$\begin{bmatrix} u' \\ y' \\ z' \end{bmatrix} = \frac{1}{g(u)} \begin{bmatrix} f_1(u, y, z) \\ f_2(u, y, z) \\ f_3(u, y, z) \end{bmatrix}, \quad (16)$$

where $g(u)$ vanishes as $u \rightarrow \pm 1$. The divergence carried by $g(u)$ can then be washed away by defining a new time variable τ , $d\tau = g^{-1}(u) dN$ [18–20], such that the previous system can be written as

$$\frac{\partial}{\partial \tau} \begin{bmatrix} u \\ y \\ z \end{bmatrix} = \begin{bmatrix} f_1(u, y, z) \\ f_2(u, y, z) \\ f_3(u, y, z) \end{bmatrix}. \quad (17)$$

¹Here, we prefer the letter s for the variable related to the CDM energy density, in detriment of the letter w used in [3].

²This compact variable was first proposed in Ref. [12] where it was identified by the letter x . To avoid a potential confusion with the nomenclature, we adopt the letter u for the compact variable, as used in Ref. [14].

The fixed points at χ -infinity can now be identified as the points $(\pm 1, y, z)$ such that $f_1 = f_2 = f_3 = 0$ [18–20]. A correct identification of these fixed points depends on whether or not the function $g(u)$ carries the divergent leading order of the equations, so that all the divergent terms are cancelled through a proper redefinition of the time variable. If this is not the case, we run the risk of “overshooting” in the divergence cancellation and introduce artificial fixed points in the system.

In [14] the proper identification and characterisation of the fixed points at χ -infinity encountered three main difficulties. First, the fact that a trigonometric, instead of a polynomial, relation was employed in Eq. (15) makes it more difficult to identify the divergence rate of the equations. Secondly, the fact that z depends on u through the potential, means that one has to take special care and understand what is the behaviour of z as $u \rightarrow \pm 1$. For potentials that vanish at infinite χ the variable z may tend to 0 sufficiently fast and cancel the divergent terms in u , e.g. the case of the Gaussian potential which was extensively discussed in Ref. [14]. In such cases, a “blind” time redefinition based only on the divergent terms in u would lead to incorrect results when identifying the fixed points of the system. Finally, even after the correct fixed points with $u = \pm 1$ were identified, their stability needed to be clarified. Within the approach we will present next, we will see how to redefine the proper conditions to use a linear stability analysis.

3.2. New dynamical system

We now present an alternative dynamical system description which tries to avoid the issues mentioned above. We begin by employing a new compactification scheme for χ

$$\frac{\chi}{\chi_c} = \frac{v}{1 - v^2}, \quad (18)$$

with $v \in [-1, 1]$. Note that the values $v = \pm 1$ correspond to $\chi \rightarrow \pm\infty$. By employing the relation (18), we ensure that all the divergent terms appear as powers of $(v \pm 1)$ with negative exponents, facilitating the identification of the leading order of the divergence.

In a second step, we decompose the variable z and isolate its explicit dependence on v . To do this, we first re-scale the potential V as

$$V(v) = \frac{3H_0^2}{\kappa^2} V_*(v), \quad (19)$$

where H_0 is the current value of the Hubble parameter and V_* is a dimensionless function of v . Next, we intro-

duce the compact Hubble rate ³

$$h := \frac{(H/H_0)^2}{1 + (H/H_0)^2}, \quad (20)$$

defined in the interval $[0, 1]$, with $h = 0$ corresponding to a Minkowski space-time, $H = 0$, and $h = 1$ to the limit ⁴ $H \rightarrow +\infty$. Using Eqs. (19) and (20) we can write the variable z as

$$z^2 = \frac{1 - h}{h} V_*(v). \quad (21)$$

We are now in a position to write the set of evolution equations for the dynamical variables (v, y, h) . From Eqs. (5), (9), (10), (18) and (21), we obtain

$$v' = 3 \frac{1 - v^2}{1 + v^2} [y(1 - v^2) - v], \quad (22)$$

$$y' = \frac{3}{2} \left\{ y(1 - y^2) - \frac{1 - h}{h} \times \left[V_*(v)y + \frac{1 - v^2}{1 + v^2} \frac{\partial V_*}{\partial v} (1 - v^2 - vy) \right] \right\}, \quad (23)$$

$$h' = -3 \left[h(1 - h)(1 - y^2) + (1 - h)^2 \left(\frac{1 - v^2}{1 + v^2} v \frac{\partial V_*}{\partial v} - V_*(v) \right) \right]. \quad (24)$$

For each type of potential, we can replace $V_*(v)$ in Eqs. (23) and (24), identify the leading divergent term in order to proceed with the appropriate time redefinition and divergence cancellation, and finally identify the fixed points at infinity and study their stability. From Eqs. (7) and (21) we can write s as

$$s^2 = 1 - y^2 - \frac{1 - h}{h} V_*(v). \quad (25)$$

To conclude this section, we look at the invariant sets \mathcal{M}_0 and \mathcal{M}_1 , cf. (14), in this new description. First, we note that the set \mathcal{M}_0 is no longer present in the system for a general potential. Instead, the behaviour of the system in \mathcal{M}_0 is given by the set of equations (22), (23) and (24) with the null potential $V_* = 0$. In the case of \mathcal{M}_1 , i.e. in the absence of CDM, the combination of Eq. (21) with the condition $z^2 + y^2 = 1$ allows us to express h as

$$h = \frac{V_*(v)}{1 - y^2 + V_*(v)}. \quad (26)$$

³A similar compact variable related to the Hubble rate is introduced in [22].

⁴We are assuming expanding cosmologies, i.e. $H \geq 0$.

As such, we can drop Eq. (24) and re-write Eqs. (22) and (23) as

$$v' = 3 \frac{1-v^2}{1+v^2} [y(1-v^2) - v], \quad (27)$$

$$y' = -\frac{3}{2} (1-y^2) \frac{1-v^2}{1+v^2} \frac{1}{V_*} \frac{\partial V_*}{\partial v} (1-v^2 - vy). \quad (28)$$

These equations are equivalent to the ones obtained in Ref. [8] in the context of 3-form inflation.

4. Gaussian potential

For concreteness, from now on we analyse the case of the Gaussian potential

$$V = \frac{3H_0^2}{\kappa^2} \bar{V} e^{-\xi \frac{\chi^2}{6\kappa^2}}, \quad (29)$$

which was studied in detail in Ref. [14]. Here, ξ and \bar{V} are positive dimensionless constants. With the ansatz (29) we can write V_* and its derivative in terms of v as

$$V_* = \bar{V} \exp \left[-\frac{\xi}{9} \left(\frac{v}{1-v^2} \right)^2 \right], \quad (30)$$

$$\frac{\partial V_*}{\partial v} = -\frac{2\xi}{9} \bar{V} \frac{(1+v^2)v}{(1-v^2)^3} \exp \left[-\frac{\xi}{9} \left(\frac{v}{1-v^2} \right)^2 \right]. \quad (31)$$

In order to completely identify the fixed points of the system at χ -infinity, we will first analyse the solutions in the subset \mathcal{M}_1 , when only a 3-form field is present, and then in the more general case, where a 3-form field and CDM are present.

4.1. FLRW with a 3-form

In this case, the set of equations that govern the evolution of the system is (27) and (28). After replacing V_* and its derivative in these equations, we find that the leading order of the divergent terms that appear in Eq. (28) is $(1-v^2)^{-2}$. We absorb these divergences by defining a new time variable τ , with $d\tau = (1+v^2)(1-v^2)^2 dN$, such that the new dynamical system reads ⁵

$$\frac{\partial v}{\partial \tau} = 3(1-v^2)^3 [y(1-v^2) - v], \quad (32)$$

$$\frac{\partial y}{\partial \tau} = \frac{\xi}{3} (1-y^2) (1-v^2 - vy) (1+v^2) v. \quad (33)$$

⁵Strictly speaking, only the factor $(1-v^2)^2$ is necessary in order to wash away the divergence of the system. The factor $(1+v^2)$ is included simply to simplify the final equations.

Fixed point	(v_{fp}, y_{fp})	h_{fp}	Phys. state	Type [3]	Type [14]
$\hat{\mathcal{B}}^\pm$	$\left(\pm \frac{\sqrt{5}-1}{2}, \pm 1 \right)$	1	LSBR	B_\pm	I
$\hat{\mathcal{C}}$	(0, 0)	$\frac{\bar{V}}{1+\bar{V}}$	dS	C	II
$\hat{\mathcal{D}}_{+1}^\pm$	$(\pm 1, +1)$	undet.	dS	n/a	III
$\hat{\mathcal{D}}_0^\pm$	$(\pm 1, 0)$	0	M	n/a	III
$\hat{\mathcal{D}}_{-1}^\pm$	$(\pm 1, -1)$	undet.	dS	n/a	III

Table 1: Fixed points of the system (32) and (33) for a cosmological model with a 3-form with a Gaussian potential (29). For each solution, we present the value of h at the fixed point, the asymptotic physical state of the Universe: LSBR - Little Sibling of the Big Rip; dS - de Sitter; M - Minkowski; and its classification according to Refs. [3] and [14].

This system has nine fixed points: $\hat{\mathcal{B}}^\pm$, $\hat{\mathcal{C}}$, $\hat{\mathcal{D}}_{+1}^\pm$, $\hat{\mathcal{D}}_0^\pm$ and $\hat{\mathcal{D}}_{-1}^\pm$. The coordinates (v_{fp}, y_{fp}) and other characteristics of the fixed points are presented in Table 1, while in Figure 2 we show the flow and the position of fixed points. Of the nine fixed points obtained, $\hat{\mathcal{B}}^\pm$ and $\hat{\mathcal{C}}$ were already identified in [8] while the other six correspond to solutions at χ -infinity: $\hat{\mathcal{D}}_{+1}^\pm$, $\hat{\mathcal{D}}_0^\pm$ and $\hat{\mathcal{D}}_{-1}^\pm$. The eigenvalues, γ_i , of the respective Jacobians are

$$\begin{aligned} \{\gamma_1, \gamma_2\}_{\hat{\mathcal{D}}_{+1}^\pm} &= \left\{ 0, \frac{4}{3}\xi \right\}, & \{\gamma_1, \gamma_2\}_{\hat{\mathcal{D}}_0^\pm} &= \left\{ 0, -\frac{2}{3}\xi \right\}, \\ \{\gamma_1, \gamma_2\}_{\hat{\mathcal{D}}_{-1}^\pm} &= \left\{ 0, \frac{4}{3}\xi \right\}. \end{aligned} \quad (34)$$

In all six cases we encounter a null eigenvalue. Therefore, to study the behaviour of the system in the vicinity of each fixed point, we cannot use the results of linear stability analysis and we must instead resort to the methods of Centre Manifold Theory (CMT) [12, 23, 24].

We begin by applying the double transformation

$$\begin{bmatrix} v \\ y \end{bmatrix} \rightarrow \begin{bmatrix} \delta v \\ \delta y \end{bmatrix} = \begin{bmatrix} v - v_{fp} \\ y - y_{fp} \end{bmatrix} \rightarrow \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = M^{-1} \begin{bmatrix} \delta v \\ \delta y \end{bmatrix} M, \quad (35)$$

where M is the square matrix whose columns are the eigenvectors of the Jacobian of the system at (v_{fp}, y_{fp}) . In the new coordinate system (X_1, X_2) , the fixed point sits at the origin (0, 0), the axes are aligned with the eigenvectors of the Jacobian, and the set of equations (27) and (28) reads

$$\frac{\partial}{\partial \tau} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \lambda_2 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} F_1(X_1, X_2) \\ F_2(X_1, X_2) \end{bmatrix} \quad (36)$$

where $F_{1,2} \sim O(\|(X_1, X_2)\|^2)$. At this point, we note that CMT [23] guarantees the existence of a centre manifold

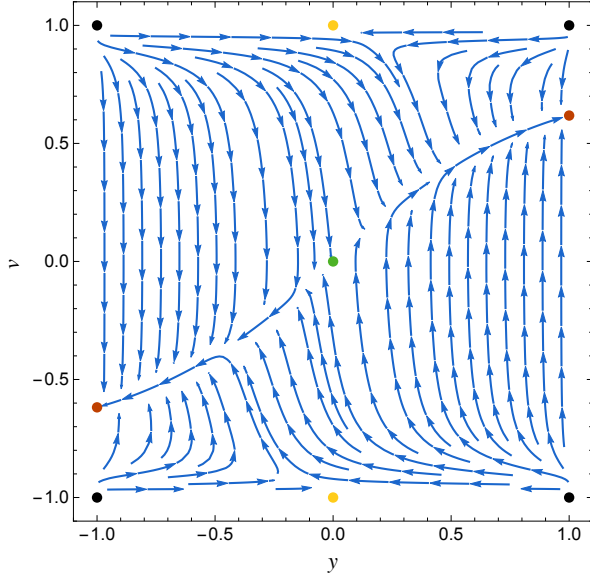


Figure 2: Flow (blue arrows) and fixed points (coloured dots) of the system (32) and (33) for a cosmological model with a 3-form with a Gaussian potential (29) with $\xi = 1$. The coordinates of the fixed points can be checked in Table 1. At χ -infinity ($v = \pm 1$) the system has six fixed points: four repulsive points and two saddles.

W^c , tangent to $\vec{e}_1 = (1, 0)$ at the origin⁶. In addition, it assures that in some neighbourhood of the origin there is a mapping $G(X_1)$, such that $(X_1, G(X_1)) \in W^c$. This mapping must satisfy $G(0) = 0$ and $(\partial G / \partial X_1)(0) = 0$, and can be Taylor expanded around $X_1 = 0$ up to an arbitrary order as

$$G(X_1) = A_2 X_1^2 + A_3 X_1^3 + \dots \quad (37)$$

The main result of CMT is that, in a vicinity of the origin, the stability of the system in W^c can be decided through the stability of the reduced system

$$\frac{\partial X_1}{\partial \tau} = F_1(X_1, G(X_1)). \quad (38)$$

We are now left with the task of finding the values of the linear coefficients A_n . For this, we combine the second row of Eq. (36) with Eq. (38) and the Leibnitz rule, $\partial X_2 / \partial \tau = (\partial G / \partial X_1)(\partial X_1 / \partial \tau)$ in order to write

$$\frac{\partial G}{\partial X_1} F_1(X_1, G(X_1)) = \lambda_2 G(X_1) + F_2(X_1, G(X_1)). \quad (39)$$

⁶In addition to the centre manifold W^c , tangent to $\vec{e}_2 = (0, 1)$ at the origin, the system has a stable manifold W_s whenever $\Re(\lambda_2) < 0$ and an unstable manifold W_u whenever $\Re(\lambda_2) > 0$. In the case of a Gaussian potential (29) with positive ξ the fixed points \hat{D}_{+1}^\pm and \hat{D}_{-1}^\pm are repulsive in the direction \vec{e}_2 while the fixed points \hat{D}_0^\pm are attractive, cf. Eqs (34).

Fixed point	(v_{fp}, y_{fp}, h_{fp})	Phys. state	Type	
			[3]	[14]
\mathcal{A}	$(0, 0, 1)$	CDM	A	I
\mathcal{B}^\pm	$\left(\pm \frac{\sqrt{5}-1}{2}, \pm 1, 1\right)$	LSBR	B_\pm	I
\mathcal{C}	$\left(0, 0, \frac{\bar{v}}{1+\bar{v}}\right)$	dS	C	II
\mathcal{E}_0^\pm	$(\pm 1, 0, 1)$	CDM	n/a	III
\mathcal{F}_{+1}^\pm	$(\pm 1, +1, h)$	dS	n/a	III
\mathcal{F}_{-1}^\pm	$(\pm 1, -1, h)$	dS	n/a	III

Table 2: Fixed points of the system (41)-(43) for a cosmological model with a 3-form with a Gaussian potential (29) and CDM. For each solution we present the asymptotic physical state of the Universe: CDM - cold dark matter domination; LSBR - Little Sibling of the Big rip; dS - de Sitter; M - Minkowski; and the classification according to Refs. [3] and [14].

This equation can be expanded up to an arbitrary order in X_1 by means of Eq. (37) and, after collecting terms of equal order in X_1 , we can iteratively compute each of the linear coefficients A_n .

When we apply this method to the fixed points \hat{D}_{+1}^\pm , \hat{D}_0^\pm and \hat{D}_{-1}^\pm , we find, for all the six cases,

$$\frac{\partial X_1}{\partial \tau} = 24X_1^3 + O(X_1^4). \quad (40)$$

Consequently, around all those fixed points the system is repulsive in the centre manifold. Since $\xi > 0$, we conclude from Eq. (34) that the two pairs \hat{D}_{+1}^\pm and \hat{D}_{-1}^\pm are repulsive and represent possible states in the asymptotic past of the system, while the pair \hat{D}_0^\pm consists of two saddle points. In addition, from Eq. (26) we find that while for \hat{D}_0^\pm the Hubble parameter vanishes at the fixed point, for the fixed points \hat{D}_{+1}^\pm and \hat{D}_{-1}^\pm the value of h is undetermined. This means that in the past, each trajectory begins in a de Sitter state as the 3-form field behaves like a cosmological constant of an arbitrary value.

The fact that the same Eq. (40) is obtained in all cases is not a coincidence. In fact, when we solve Eq. (40) to leading order and re-write the solution in terms of χ , we conclude that for very large values of $|\chi|$ the field χ behaves as $\chi \sim a^{-3}$. This result is in concordance with the behaviour obtained for $|\chi| \gg \chi_c$ from the analysis of the Friedmann equation, cf. Ref. [14], and with the tracking/scaling behaviour discussed in Ref. [4].

4.2. FLRW with a 3-form and CDM

We now focus our attention on cosmological models with CDM and a 3-form field with a Gaussian potential.

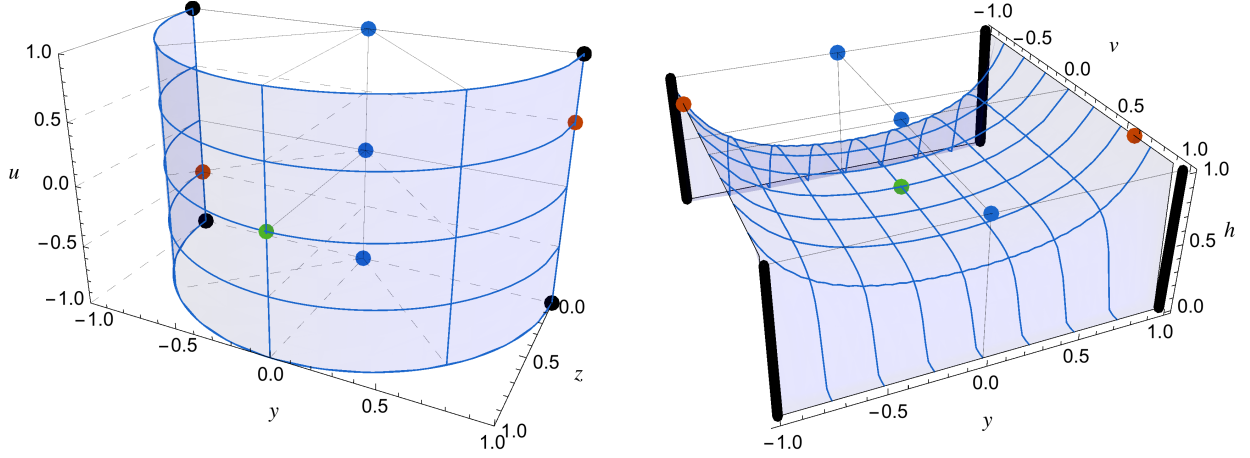


Figure 3: Position of the fixed points of the cosmological model with a 3-form field with a Gaussian potential (see Eq. (29) with $\xi = \bar{V} = 1$) and CDM in: (left panel) the dynamical system description (u, y, z) of Refs. [12, 14]; (right panel) the dynamical system description (v, y, h) employed in this work. Corresponding fixed points are signalled with the same colour in both panels. In each panel the subset \mathcal{M}_I corresponding to the case of no CDM is identified by the blue surface. The Friedmann constraint imposes that the physical system evolves within the half-cylinder on the left panel and above the surface \mathcal{M}_I on the right panel. The coordinates (v_{fp}, y_{fp}, h_{fp}) of each point are presented in Table 2.

By substituting (30) and (31) in Eqs. (22), (23) and (24) we obtain

$$v' = 3 \frac{1-v^2}{1+v^2} [y(1-v^2) - v], \quad (41)$$

$$y' = \frac{3}{2} \left\{ y(1-y^2) - \bar{V} \frac{1-h}{h} \exp \left[-\frac{\xi}{9} \left(\frac{v}{1-v^2} \right)^2 \right] \right. \\ \left. \times \left[y - \frac{2\xi}{9} \frac{v}{(1-v^2)^2} (1-v^2 - vy) \right] \right\}, \quad (42)$$

$$h' = -3 \left\{ h(1-h)(1-y^2) + \bar{V}(1-h)^2 \right. \\ \left. \times \exp \left[-\frac{\xi}{9} \left(\frac{v}{1-v^2} \right)^2 \right] \left[-\frac{2\xi}{9} \frac{v^2}{(1-v^2)^2} - 1 \right] \right\}, \quad (43)$$

with the constraint, derived from Eq. (25),

$$0 \leq (1-y^2)h - (1-h)V_*(v) \leq 1. \quad (44)$$

Due to the fast decay of the potential, all terms in Eqs. (23) and (24) which depend on the quadratic exponential vanish for $v \rightarrow \pm 1$ and no v -divergences exist in the system. In addition, we note that in the presence of CDM and for the Gaussian potential (29) with positive ξ , the Hubble parameter never vanishes, i.e., $0 < h \leq 1$.

By inspection of Eqs. (22)-(24), we find that the system has six isolated fixed points: \mathcal{A} , \mathcal{B}^\pm , \mathcal{C} and \mathcal{E}_0^\pm ; and four sets of non-isolated fixed points: \mathcal{F}_{+1}^\pm and \mathcal{F}_{-1}^\pm (cf. Table 2 and the right panel of Figure 3). We will focus our attention on the stability of the system near the

fixed points \mathcal{E}_0^\pm , \mathcal{F}_{+1}^\pm and \mathcal{F}_{-1}^\pm , since these are the ones that correspond to states of the system at χ -infinity.

For the pair \mathcal{E}^\pm , the Jacobian has three positive eigenvalues, γ_i ,

$$\{\gamma_1, \gamma_2, \gamma_3\}_{\mathcal{E}^\pm} = \left\{ \frac{3}{2}, 3, 3 \right\}. \quad (45)$$

indicating that the points are repulsive and represent possible states in the asymptotic past of the system. From Eq. (25), we obtain $s_{fp}^2 = 1$, therefore these points represent a past matter era.

Alternatively, for all the fixed points belonging to the sets $\mathcal{F}_{+1}^\pm = (1, 1, h)$, $\mathcal{F}_{+1}^- = (1, -1, h)$, $\mathcal{F}_{-1}^+ = (-1, 1, h)$ and $\mathcal{F}_{-1}^- = (-1, -1, h)$, with $0 < h \leq 1$, the eigenvalues of the Jacobian are

$$\{\gamma_1, \gamma_2, \gamma_3\}_{\mathcal{F}_{\pm 1}^\pm} = \{3, -3, 0\}. \quad (46)$$

The null eigenvalue corresponds to the direction tangent to the line of fixed points. Since the other two eigenvalues have non-zero real part, every point in the sets $\mathcal{F}_{\pm 1}^\pm$ is said to be normally hyperbolic [7, 25] and the stability of the trajectories along the remaining directions can be determined by applying the usual linear stability theory. The existence of a positive and a negative eigenvalues lead us to conclude that the sets $\mathcal{F}_{\pm 1}^\pm$ are composed entirely by saddle points which are neither attractive nor repulsive. A comparison with the results in Sec. 4.1 shows that the sets $\mathcal{F}_{\pm 1}^\pm$ correspond to the fixed points $\hat{\mathcal{D}}_{\pm 1}^\pm$ of Table 1.

5. Discussion and Conclusions

In our previous paper [14], we employed a dynamical system approach to study late-time cosmologies with CDM and DE, the latter modelled by a massive 3-form field, and we identified six new fixed points which were previously disregarded in the literature. These new solutions correspond to states of the system in the limit of infinite values of the 3-form field. In the present work, we further explore the mathematical tools necessary for analysing the flow of a system in the vicinity of infinity. Through the use of a new dynamical system description and compactification scheme [16–20], we were able to unequivocally identify the fixed points of the system in a model with a 3-form with a Gaussian potential, both in the presence and absence of non-interacting CDM. Subsequently, using linear stability theory, Centre Manifold Theory [12, 23, 24], and the notion of normally hyperbolic fixed points [7, 25], we deduced the stability of the fixed points obtained.

In absence of CDM, we found six fixed points at χ -infinity (cf. Table 1), of which two are saddles and four are repulsive. The latter correspond to the past asymptotic state of the system and represent an initial de Sitter inflationary epoch. The value of the Hubble parameter during this inflationary era is not determined by the system, instead it is characteristic of each individual trajectory.

Alternatively, in the presence of CDM we obtained a pair of isolated fixed points, \mathcal{E}^\pm , and four connected sets, $\mathcal{F}_{\pm 1}^\pm$, of non-isolated fixed points corresponding to states at χ -infinity (cf. Table 2). A detailed stability analysis shows that the two isolated solutions are repulsive in nature and correspond to matter dominated eras in the asymptotic past, while the four sets $\mathcal{F}_{\pm 1}^\pm$ are composed of normally hyperbolic saddle points. These results corroborate the findings of Ref. [14], where two repulsive, π_0^\pm , and four saddle, $\pi_{\pm 1}^\pm$, fixed points were found at χ -infinity. The difference in dimensionality of the manifolds of equilibria π_0^\pm and $\mathcal{F}_{\pm 1}^\pm$ can be explained by noting that the dynamical system employed in Ref. [14] does not track the value of the Hubble parameter and therefore does not distinguish between points with different values of h . We note that the method presented in this paper allows us to compute the eigenvalues of the fixed points at χ -infinity while avoiding the difficulties encountered in Ref. [14].

Although the results presented here are relative to the case of a 3-form with a Gaussian potential, the set of variables (v, y, h) introduced in this work can be easily applied to other types of potentials, with the exception of potentials with zeros at finite values of χ , as the

dynamical system (22)–(24), is not appropriate to study fixed-points with $h = 0$. This limitation is reminiscent of the fact that the original system (x, y, z) is not well defined for such potentials [3]. In addition, the fact that a separate compactification strategy is employed for each degree of freedom may generate an unclear interpretation when studying trajectories where more than one degree of freedom diverges. A way to overcome both these inconvenient situations is to go back to the original degrees of freedom of the model $(H, \rho_m, \chi, \dot{\chi})$ and apply a compactification scheme of the kind discussed in [18–20]. In principle, this would allow for a complete dynamical system description of cosmological models with 3-forms that could be applied to a general potential. Finally, we note that such an approach could be extended to models with a scalar field which, contrary to a 3-form, can have fixed points at infinite values of the scalar field even in the asymptotic future. This will be explored in a future work [26].

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